ON KOTZIG'S NIM

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Received: 5/16/11, Revised: 1/5/14, Accepted: 6/29/14, Published: 9/17/14

Abstract
In 1946, Anton Kotzig introduced Kotzig's Nim, also known as Modular Nim. This impartial, combinatorial game is played by two players who take turns moving around a circular board; each move is chosen from a common set of allowable step sizes. We consider Kotzig's Nim with two allowable step sizes. Although Kotzig's Nim is easy to learn and has been known for many decades, very few theorems have been established. We prove a new primitive theorem about Kotzig's Nim. We also introduce a series of conjectures about periodicities in Kotzig's Nim, based on computational exploration of the game. We share a free database of our computation, to entice others into further explorations of Kotzig's Nim. We hope the present paper sparks a revival of interest in this enticing game, for which the winning strategy (in most cases) remains an enigma.

–Dedicated to the memory of Philippe Flajolet.

1. Introduction

Guy and Nowakowski's Unsolved Problems in Combinatorial Games [3] asks, "**A6 (17).** Extend the analysis of Kotzig's Nim (WW, 515–517). Is the game eventually periodic in terms of the length of the circle for every finite move set? Analyze the misère version of Kotzig's Nim."

Kotzig's Nim [1], also known as *Modular Nim* [2], is a combinatorial game, i.e., it has two players who each have perfect information about the game and no chance moves. Kotzig's Nim is named after its inventor, Anton Kotzig [4]. The game is played on a circle with $n \in \mathbb{N}$ locations, numbered clockwise from 0 through $n-1$. The same set of potential moves $M = \{m_1, \ldots, m_k\} \in \mathbb{N}^k$ is available to each player, and thus the game is impartial. Players alternate moves, always choosing a move

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$m \in M$ and moving exactly m units in a clockwise direction on the board. A player cannot land on a previously occupied location. For this reason, the player places a mark at the current board location immediately upon arrival, so that it will never be revisited during the remainder of the game. The game ends when a player is completely obstructed, i.e., a player discovers that every set of potential moves in M leads to a previously occupied location; such an unfortunate player loses in the normal style of play.

Kotzig's Nim is a special case of the general problem known as Generalized Geography (`http://en.wikipedia.org/wiki/Generalized_geography`) which was proved by Thomas J. Schaefer to be PSPACE-complete [5].

The two most recent discussions of Kotzig's Nim [2, 1] have a difference in the way that the game begins. In [1], the board is assumed to be empty at the start of the game, so Player I begins the game by moving to any location at the start. Without loss of generality (i.e., by a possible circular shift of the labels), Player I begins the game at location 0. Subsequently, Player II moves to a location m_{i_1} mod n, and Player I moves to $(m_{i_1} + m_{i_2})$ mod n, etc. In [2], location 0 is already labeled at the start, so Player I begins by moving to a location m_{i_1} mod n at the start. Subsequently, Player II moves to $(m_{i_1} + m_{i_2})$ mod n, etc. By pretending that Player II visited location 0 at the start in [2], the two interpretations of the game are seen to be exactly the same, *except that the roles of the two players are reversed.* In other words, Player I in [1] corresponds precisely to Player II in [2].

With the normal play rule, P- and N-positions are defined in the standard recursive way:

- A P-position is any position from which the Previous player can force a win. An N position is any position from which the Next player who moves can force a win. The sets of P- and N-positions are denoted, respectively, by $\mathcal{P}$ and $\mathcal{N}$.
- All terminal positions are P-positions.
- From every N-position, there is at least one move to a P-position.
- From every P-position, every move is to an N-position.

Since the roles of Players I and II are exactly reversed in [1] and [2], then the N and P positions are exactly reversed too, i.e., the set of P positions in [1] are exactly the set of N positions in [2].

We adhere to the conventions of [2], in particular, to the convention that location 0 is already labeled at the start of the game, Player I moves to a location m_{i_1} mod n, etc.

The notational convention of [2] is $\Gamma = (m_1, \ldots, m_k; n) \in \mathcal{N}$ or $\in \mathcal{P}$, if, respectively, the first or second player can force a win from the start of a game of Kotzig's

Nim with move set $M = \{m_1, \ldots, m_k\}$ and board size n. We simplify this notation slightly (dropping the equals sign), by writing $\Gamma(m_1, \ldots, m_k; n) \in \mathcal{N}$ or $\in \mathcal{P}$.

2. An Example

As an example of play in Kotzig's Nim, we consider the game with move set $M = \{1, 2\}$ and $n = 7$. This example is discussed on page 516 of [1], where the solution is given as a table. Instead, we present the solution for this example as a tree. The first player can move from position 0 to either position 1 or position 2. If the first player moves to position 1, then the game is finished after a total of four moves. If the first player moves to position 2, then the game is finished after a total of six moves. In either case, the second player has a winning strategy. The tree associated with the game play for this example is displayed in Figure 1. For ease of interpretation, *we use solid lines for the first player's moves and dashed lines for the second player's moves.* We indicate when the *losing player's* move is "forced" (but we never call the winning player's moves "forced").

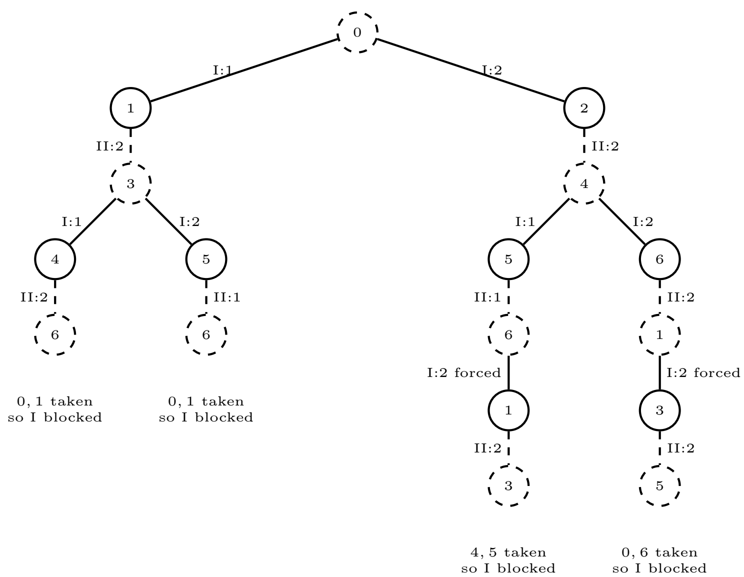

Figure 1: The tree associated with the Example in Section 2. It shows the winning strategy for Player II in Kotzig's Nim $\Gamma(1, 2; 7)$. This is a "reduced" version of the full game tree. It indicates the moves that the winning player should take. (It omits the winning player's other possible moves.)

3. A New Primitive Theorem

In [2], Corollary 2, Theorem 7, and Theorem 8 completely classify (respectively) the games $\Gamma(1,2;n)$, $\Gamma(2,3;n)$, and almost $\Gamma(3,4;n)$ ("almost" because the cases $n = \pm 3 \bmod 7$ are only conjectures in [2]). We emphasize that $m_2 - m_1 = 1$ in each case.

The argument in p. 516–517 of [1], attributed to Nowakowski, also classifies $\Gamma(1,3;n)$. Here, $m_2 - m_1 = 2$.

We are unaware of any proofs in which the complete classification of $\Gamma(1,4;n)$ is given. Since $m_2 - m_1 = 3$, this turned out to be a quite challenging task, but we give a complete classification and proof in Theorem 3.1. This is the main achievement of the present paper. In Section 4, we pose many new open questions that the reader may wish to analyze.

Theorem 3.1. *If* $n \in \{1,3,5,7,15\}$*, or if* $n \equiv 3 \bmod 5$ *and* $n \geq 23$*, then* $\Gamma(1,4;n) \in \mathcal{P}$*; otherwise,* $\Gamma(1,4;n) \in \mathcal{N}$*.*

Proof. Considering the value of n modulo 5, there are five cases that we consider. We provide tree diagrams to directly establish each of the five cases. In the tree diagrams, the solid lines and circles correspond to moves by Player I, and the dashed lines and circles correspond to moves by Player II. We use a notation on some of the figures, in a red font, to indicate which positions on the board have been visited. The diamonds correspond to the "diamond strategy", in which the player with the winning strategy chooses the opposite type of move chosen by the other player. For example, in Figure 3, on the right-hand side of the tree, Player II has recently moved from 4 to 5. If Player I moves 1 to land at 6, then Player II does the opposite type of move, i.e., moves 4, to land at 3; on the other hand, if Player I moves 4 to land at 2, then Player II does the opposite type of move, i.e., moves 1, to land at 3. For more examples of the diamond strategy, see [2].

`Case I.` $n = 5k+1$. If $n = 1$, then Player I is immediately stuck, and thus $\Gamma(1,4;1) \in \mathcal{P}$. Otherwise, $\Gamma(1,4;n) \in \mathcal{N}$. For $k \geq 1$, Figure 2 shows the game tree that describes a winning strategy for the first player.

Player I should start with 1.

Each time Player II wants to move 4, Player I responds by moving 1. If Player II makes k consecutive moves of 4, then Player I should respond by consistently moving 1 during the first $k-1$ responses, and then Player I should move 4 as the kth response. This is the start of the right-hand side of the tree in Figure 2. Then the diamond strategy will be used $k-1$ times in a row, and Player II is forced into a loss.

On the other hand, if Player II moves 4 exactly j consecutive times for $0 \leq j \leq k-1$, and then switches to moving 1 on his $(j+1)$st move, Player I should respond by consistently moving 1 for the first $j+1$ responses. This is the start of the left-

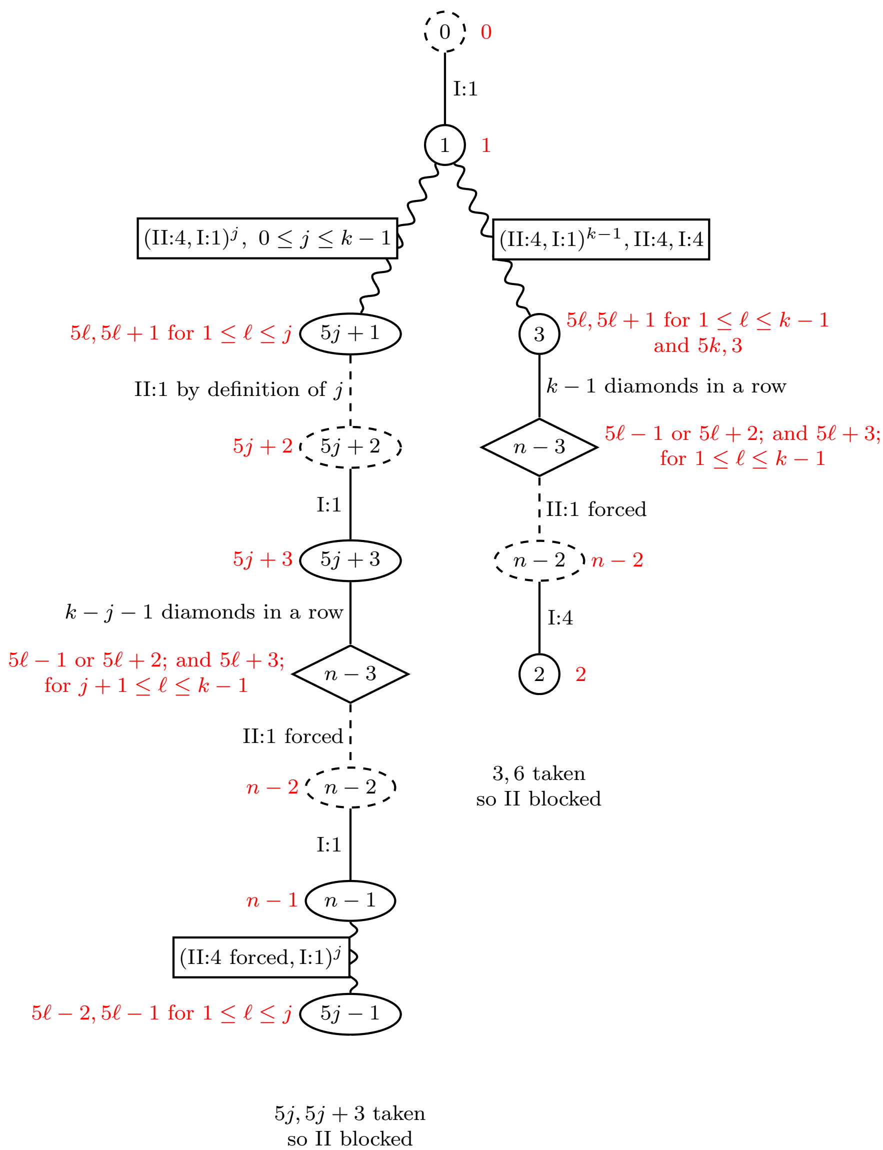

Figure 2: Winning strategy for Player I for $\Gamma(1,4;5k+1)$, where $k \ge 1$ and $n = 5k+1$.

hand side of the tree in Figure 2. Then $k-j-1$ uses of the diamond strategy force Player II into a loss.

`Case II.` $n = 5k+2$. If $n = 2$, then Player I moves 1 and then Player II is stuck, and thus $\Gamma(1,4;2) \in \mathcal{N}$. If $n = 7$, then Player II has the winning strategy shown in Figure 3, so $\Gamma(1,4;7) \in \mathcal{P}$. Otherwise, for $n = 5k+2$, with $k \geq 2$, we have $\Gamma(1,4;n) \in \mathcal{N}$; Figure 4 shows the game tree that describes a winning strategy for the first player.

Player I should start with 4.

If Player II moves 1, the play proceeds as on the left of the tree in Figure 4, from position 5.

If Player II moves 4 strictly less than k times in a row (say, j times in a row), then Player I should respond by consistently moving 1. Then, after Player II switches to move 1 on his $(j+1)$st move, Player I should respond by moving 1 again. After a series of $k-j-1$ diamonds, Player I can force the remainder of the moves. This is the middle of the tree in Figure 4.

If Player II moves 4 at least k times in a row, Player I should respond by consistently moving 1 during his first $k-1$ responses, and then switch to moving 4 for his kth response. Afterwards, Player I wins by forcing the remainder of the moves. This is the right-hand side of the tree in Figure 4.

`Case III.` $n \equiv 3 \bmod 5$. This case is quite different because, for large n, Player II has a winning strategy. If $n = 3$, each player moves once, then Player I is stuck, so Player II wins. If $n = 8, 13, 18$, then Player I has the winning strategies in Figure 5a, Figure 5b, Figure 6, respectively, so $\Gamma(1,4;n) \in \mathcal{N}$ for $n = 8, 13, 18$. For $n \equiv 3 \bmod 5$, with $n \geq 23$, $\Gamma(1,4;n) \in \mathcal{P}$; Figures 7, 8, 9, 10, 11, show the winning strategy for Player II.

`Case IV.` $n = 5k+4$. We have $\Gamma(1,4;n) \in \mathcal{N}$. For $k \geq 0$, Figure 12 displays the tree corresponding to a winning strategy for player I.

Player I should start with 1. Let j (for $0 \leq j \leq k-1$) denote the number of times at the start that Player II chooses to make a move of 4, before switching to a move of 1 on the $(j+1)$st; Player I should always follow with a move of 1. If, on the other hand, Player II makes k consecutive moves of 4—and if Player I responds with a move of 1 each time—then Player II will be forced to switch to 1 on Player II's $(k+1)$st move; thus, naturally $j = k$ in this case.

Afterwards, Player I should move from $5j+2$ to $5j+3$, and then should implement $k-j$ diamonds in a row. The remaining moves of Player II are forced. This is depicted in Figure 12.

`Case V.` $n = 5k$. If $n = 5$ or $n = 15$, then Player II has the winning strategy shown in Figures 13a and 14, so $\Gamma(1,4;5) \in \mathcal{P}$ and $\Gamma(1,4;15) \in \mathcal{P}$. Also, $\Gamma(1,4;10) \in \mathcal{N}$ as in Figure 13b.

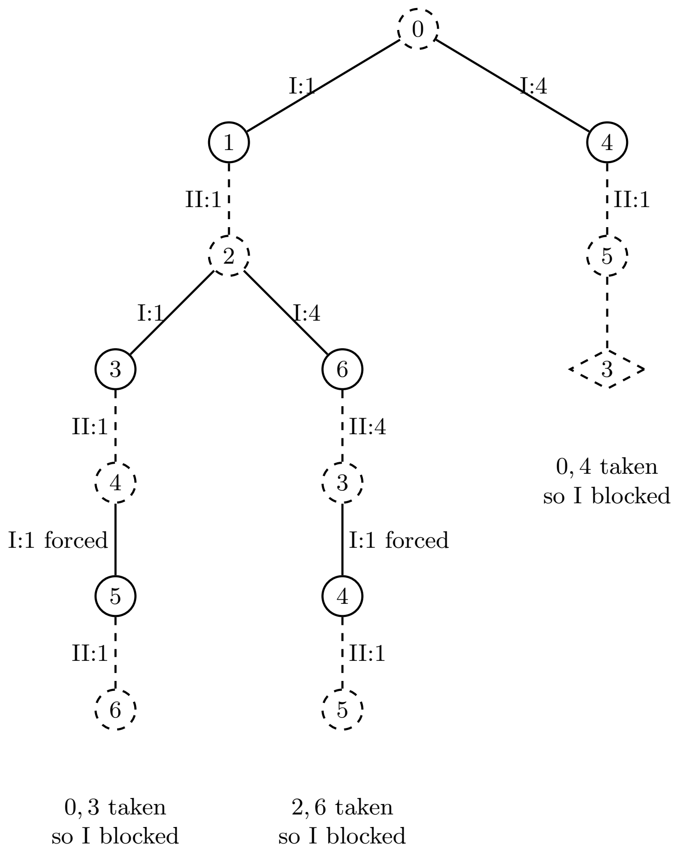

Figure 3: Winning strategy for Player II for $\Gamma(1, 4; 7)$.

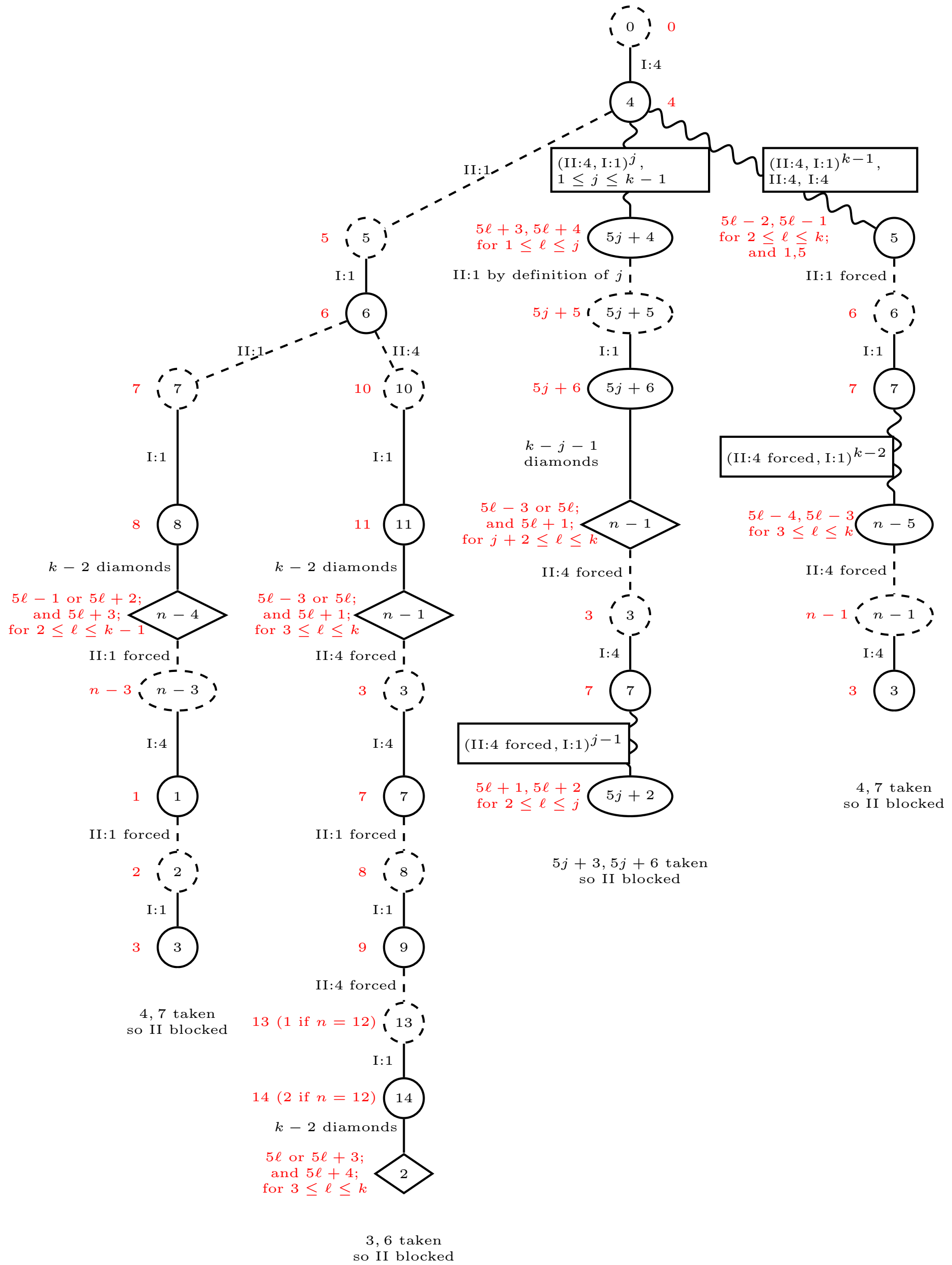

Figure 4: Winning strategy for Player I for $\Gamma(1, 4; 5k+2)$, where $k \ge 2$ and $n = 5k+2$.

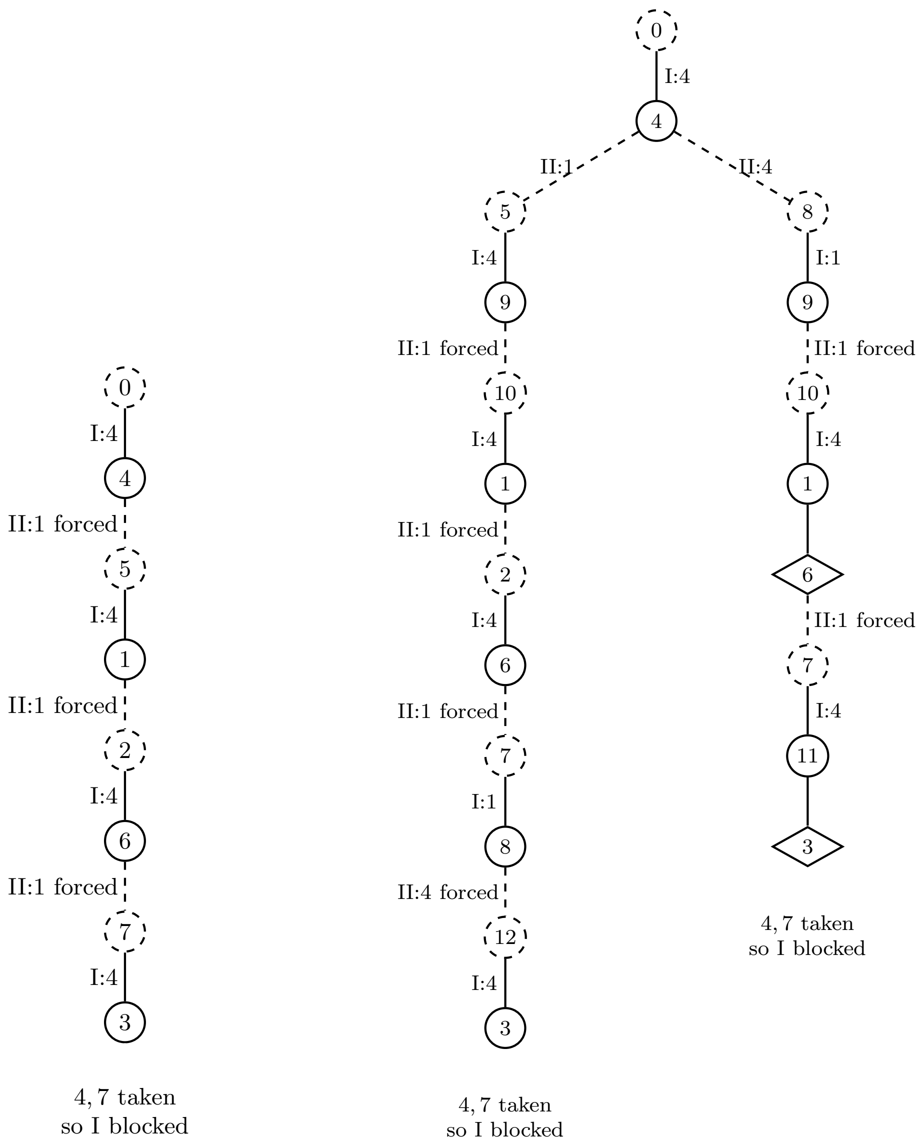

Figure 5: Winning strategy for Player I for (a.) $\Gamma(1, 4; 8)$; (b.) $\Gamma(1, 4; 13)$.

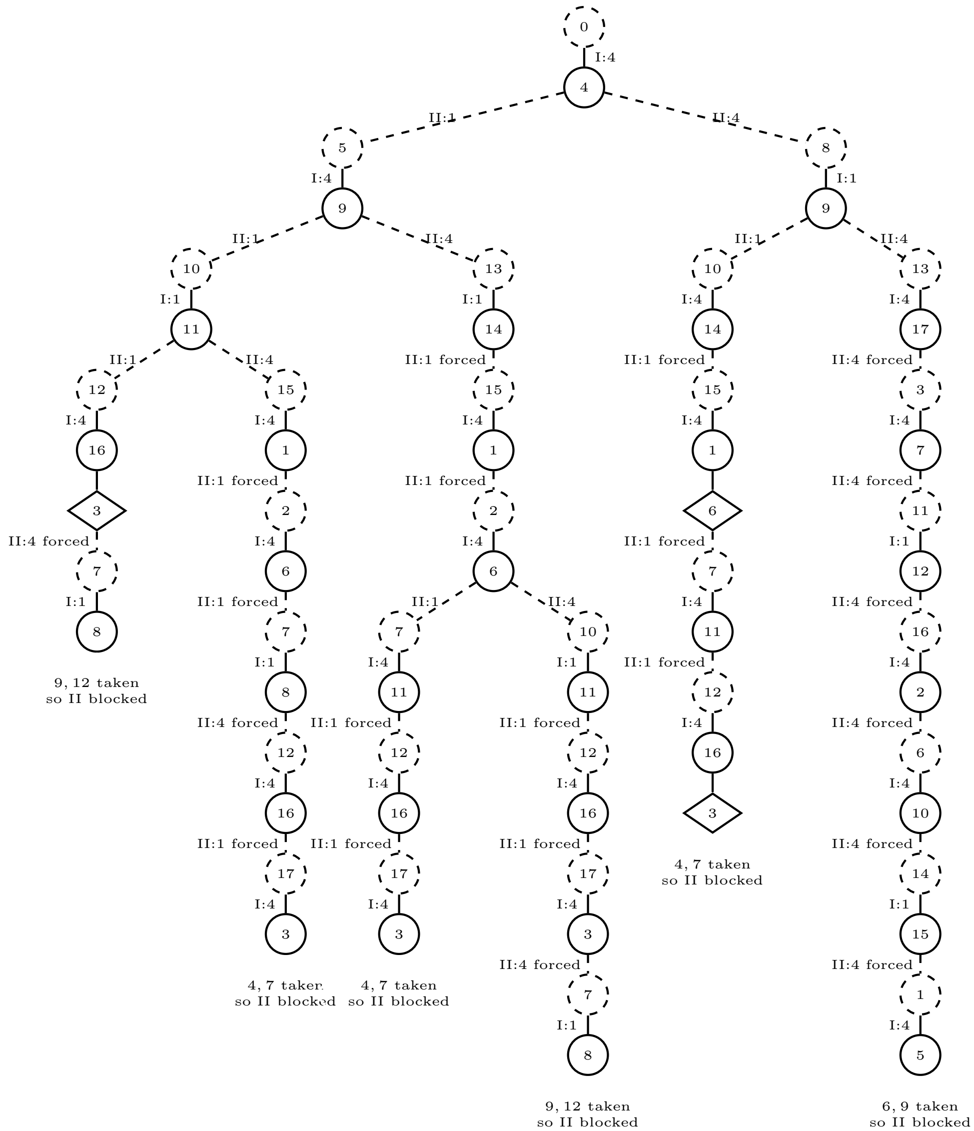

Figure 6: Winning strategy for Player I for $\Gamma(1, 4; 18)$.

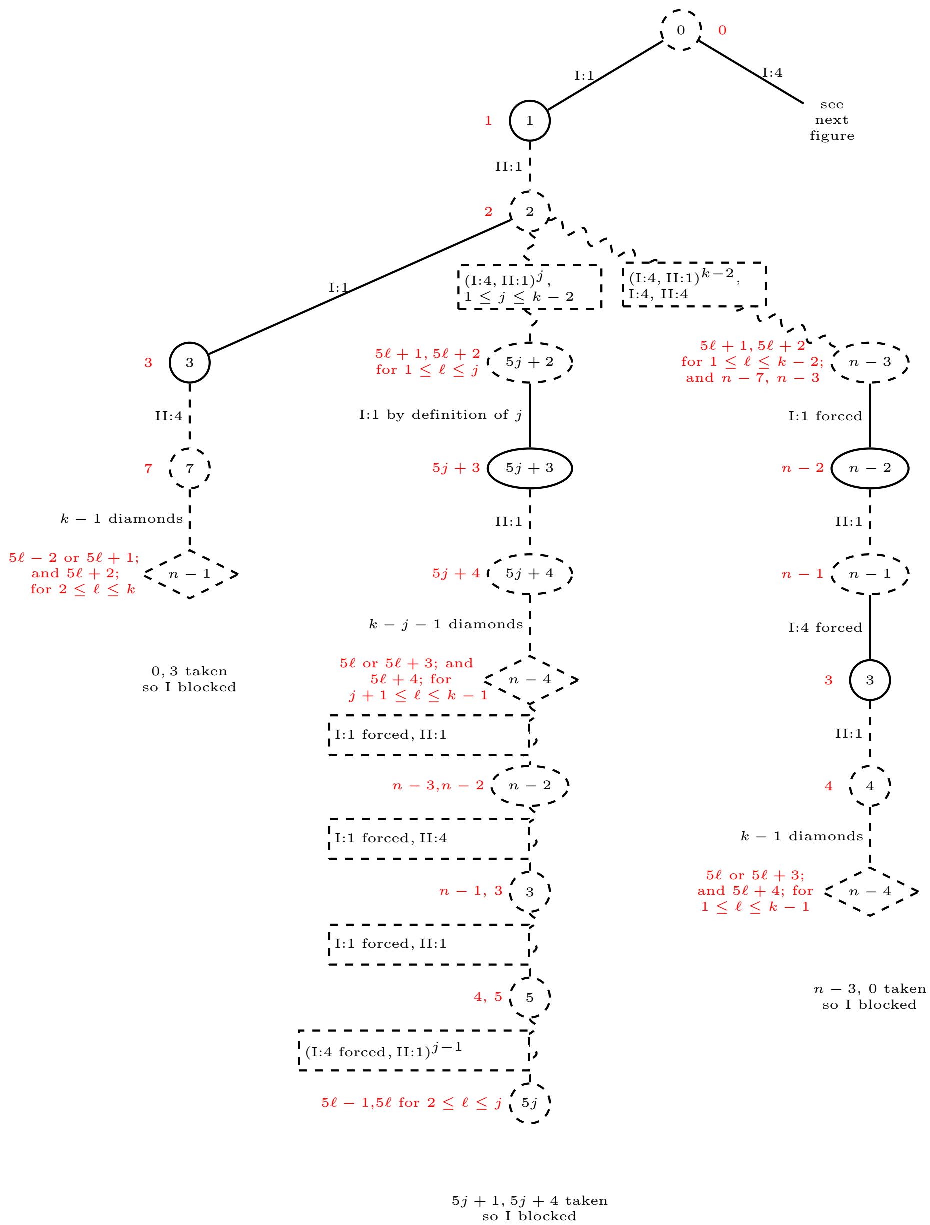

Figure 7: Winning strategy for Player II for $\Gamma(1, 4; 5k+3)$, where $k \geq 4$ and $n = 5k+3$.

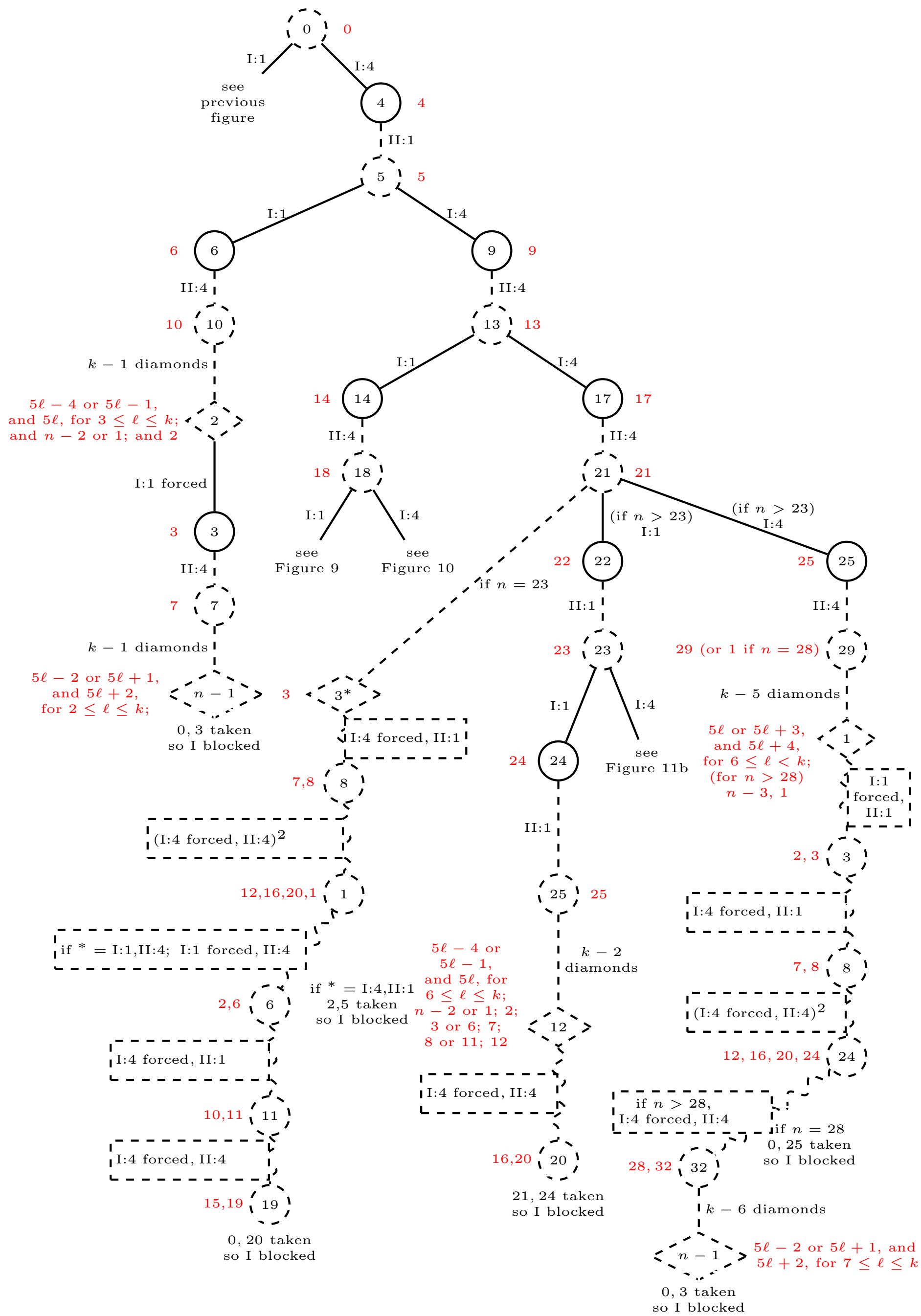

Figure 8: Winning strategy for Player II for $\Gamma(1, 4; 5k + 3)$, where $k \geq 4$ and $n = 5k + 3$.

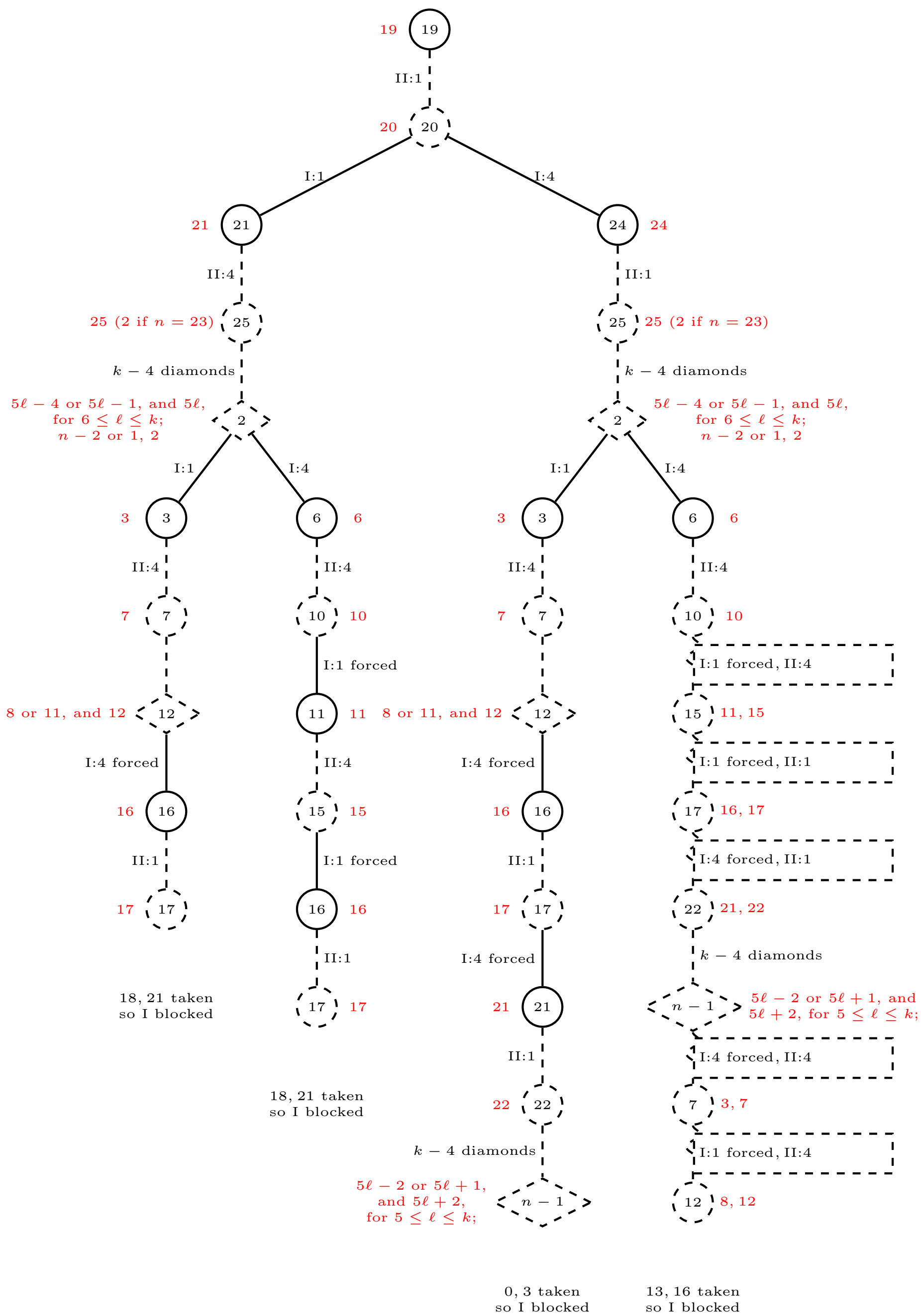

Figure 9: Winning strategy for Player II for $\Gamma(1, 4; 5k + 3)$, where $k \geq 4$ and $n = 5k + 3$.

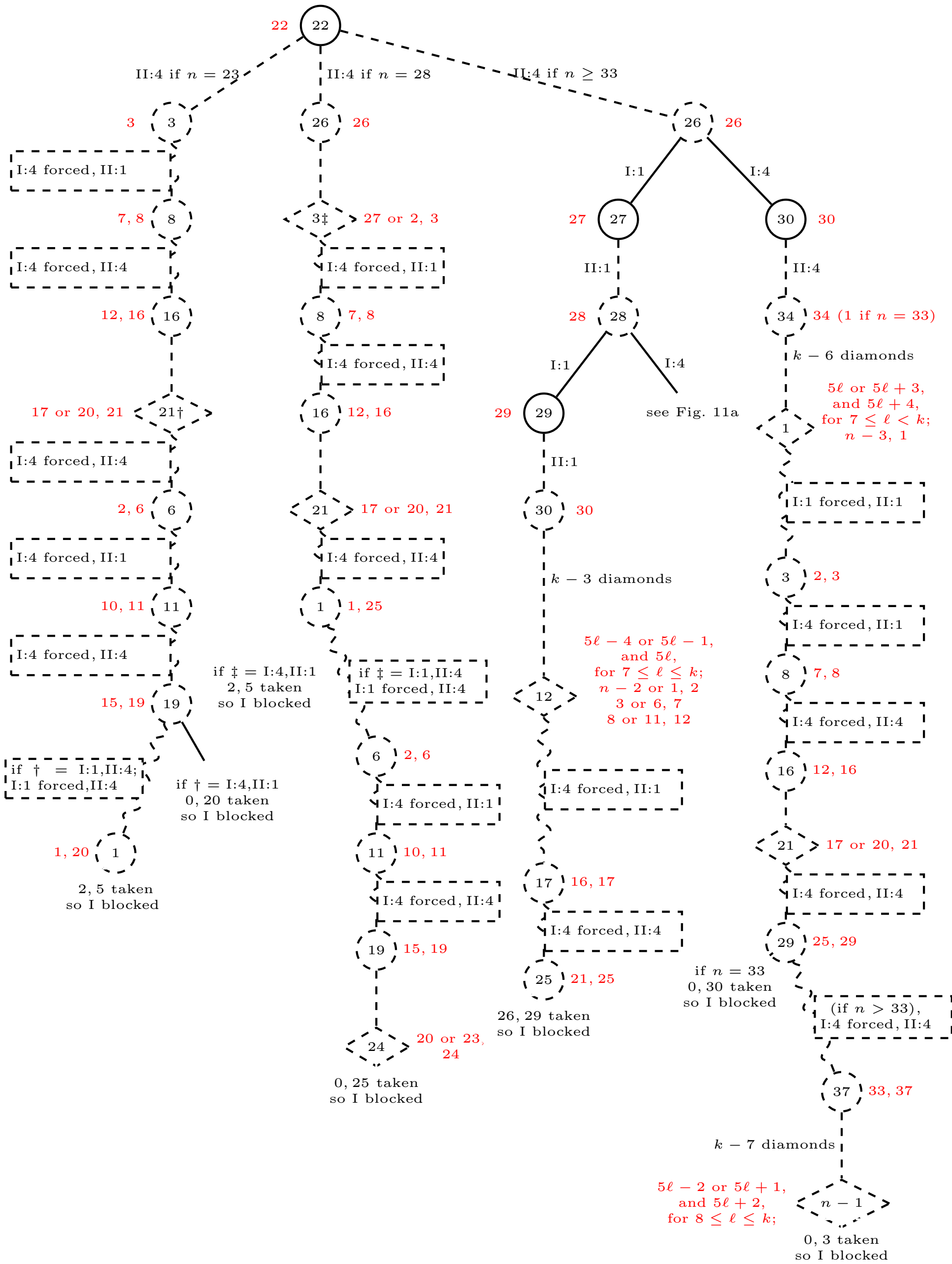

Figure 10: Winning strategy for Player II for $\Gamma(1, 4; 5k + 3)$, where $k \geq 4$ and $n = 5k + 3$.

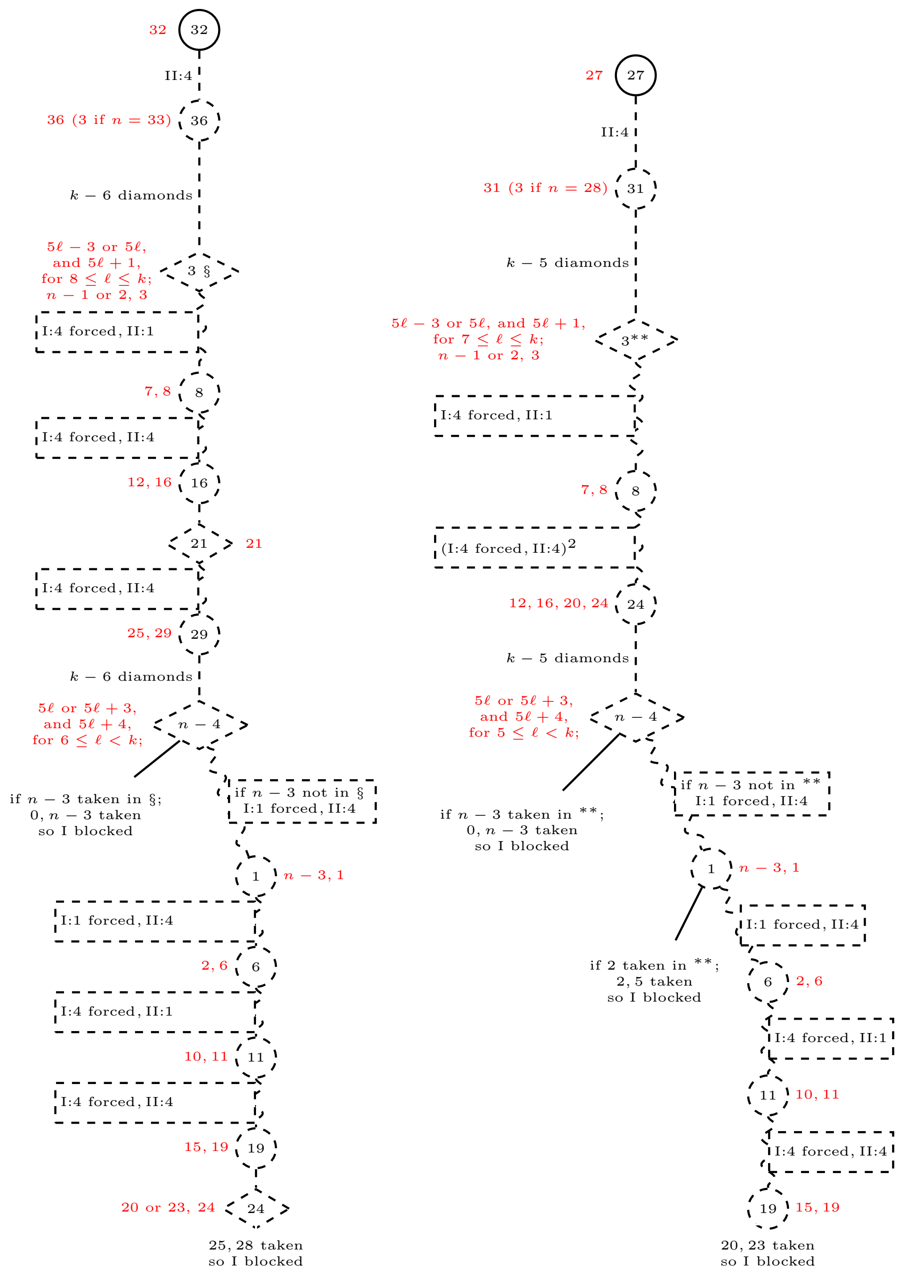

Figure 11: Winning strategy for Player II for $\Gamma(1,4;5k+3)$, where $k \ge 4$ and $n = 5k+3$; the left-hand side is Figure 11a; the right side is Figure 11b.

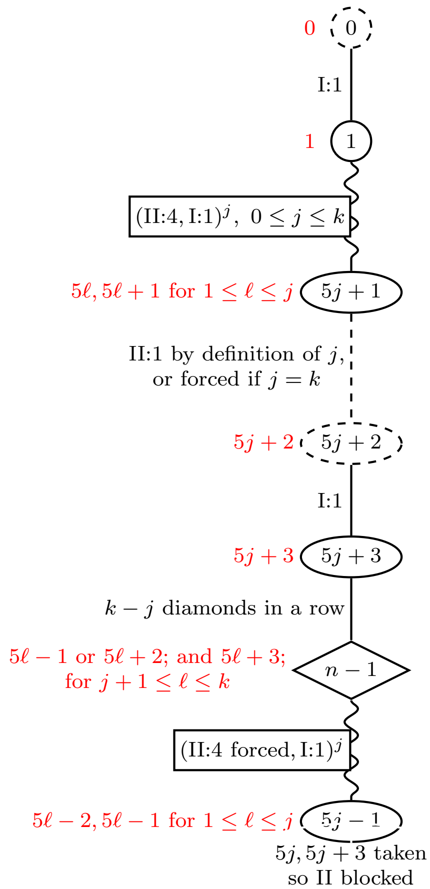

Figure 12: Winning strategy for Player I for $\Gamma(1, 4; 5k + 4)$, where $k \ge 1$ and $n = 5k + 4$.

Otherwise, for $n \equiv 0 \bmod 5$, with $n \geq 20$, we have $\Gamma(1, 4; n) \in \mathcal{N}$; Figures 15 and 16 show the winning strategy for Player I in these cases. Player I should move 4 at the start.

If Player II makes a move of 1, to position 5: Player I should move 1, and then the play splits according to the left-hand side of Figure 15: If Player II makes a move of 1, then Player I moves 1 also, and a series of $k-1$ diamonds ends the game. On the other hand, if Player II makes a move of 4, then Player I moves 4 too, followed by $k-3$ diamonds. Then Player II moves 4, Player I moves 4, and the game ends with $k-1$ diamonds.

If Player II makes a move of 4, to position 8, then Player I should move 4, to position 12. The rest of the situation is described on the right-hand side of Figure 15 and throughout Figure 16. We break the game into cases:

`(A.)` Right-hand side of Figure 15. Suppose Player II moves 1; then Player I should move 1. If Player II moves 1, then Player I moves 1, and a series of $k-1$ diamonds ends the game. On the other hand, if Player II moves 4, then Player I moves 1 and uses $k-4$ diamonds to arrive at position $n-1$. Then Player II's next three moves are forced, followed by Player I moving the same way each time (4, 4; 4, 4; 1, 1). Then, after a series of $k-3$ diamonds, Player II can make just one more move and then Player I forces the end.

`(B.)` Middle of Figure 16. If Player II moves 4 a total of j (for $j \leq k-3$) times, then Player I responds by moving 1 each time. Then (by definition) Player II moves 1, Player I moves 1, and then Player I uses a series of $k-j-3$ diamonds. Finally, Player II is forced to move 4 twice, with Player I responding with a move of 4 each time, and then a series of $j-1$ pairs of moves (Player II moving 4, Player I moving 1) ends the game.

`(C.)` Right-hand side of Figure 16. If Player II moves 4 a total of $k-2$ times, Player I should respond by moving 1 during the first $k-3$ times, and then move 4 on the $(k-2)$nd time. Then the play varies, according to whether Player II moves 1 or 4:

`(C part 1.)` If Player II moves 1, to position 6, then Player I moves 1, Player II is forced to move 4, and then Player I moves 4. Player II is forced to move 4 for $k-4$ times, followed by Player I moving 1 each time. Then Player II must move 4 again, and Player I moves 4 to end the game.

`(C part 2.)` If Player II moves 4, to position 9, then Player I moves 4, Player II is forced to move 1, and then Player I moves 1. Player II is forced to move 4 for $k-4$ times, followed by Player I moving 1 each time. Then Player II must move 4 twice, with Player I responding by moving 4 each time, and the game ends.

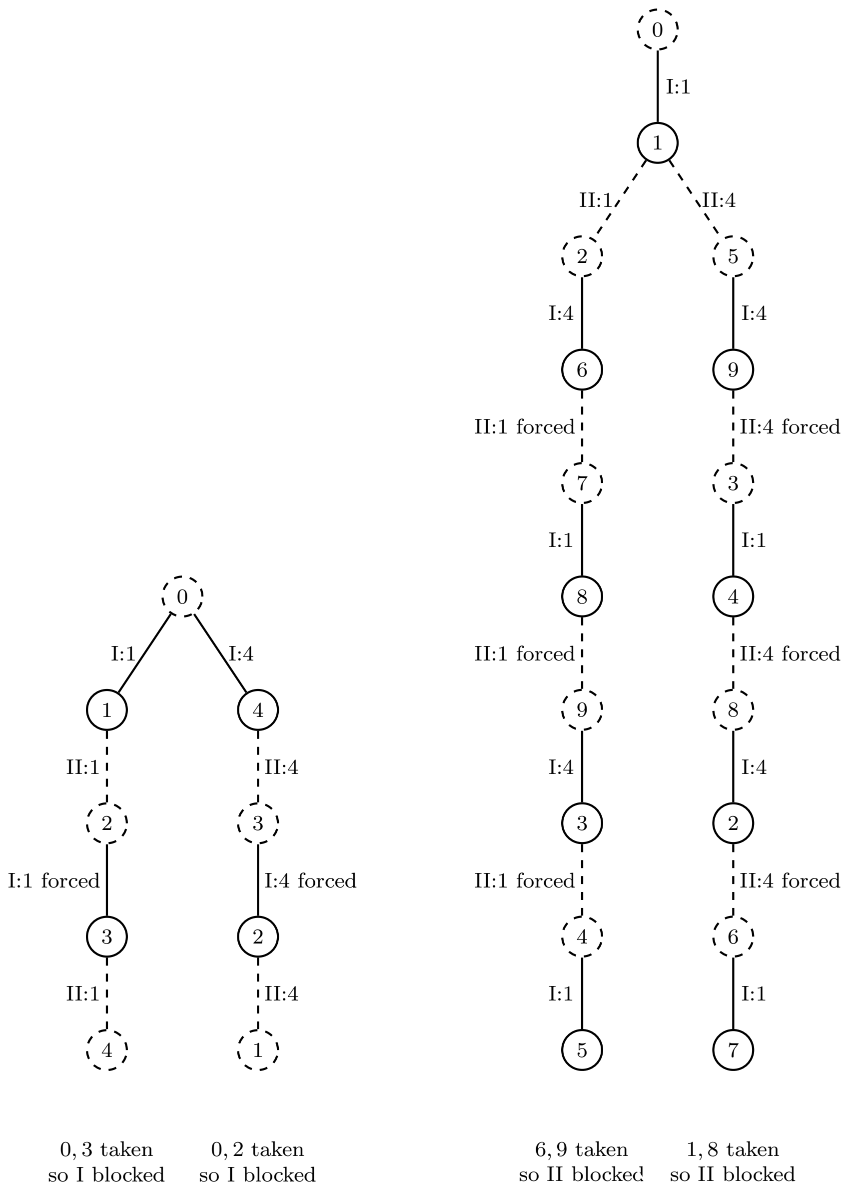

Figure 13: (a.) Winning strategy for Player II for $\Gamma(1, 4; 5)$; (b.) Winning strategy for Player I for $\Gamma(1, 4; 10)$.

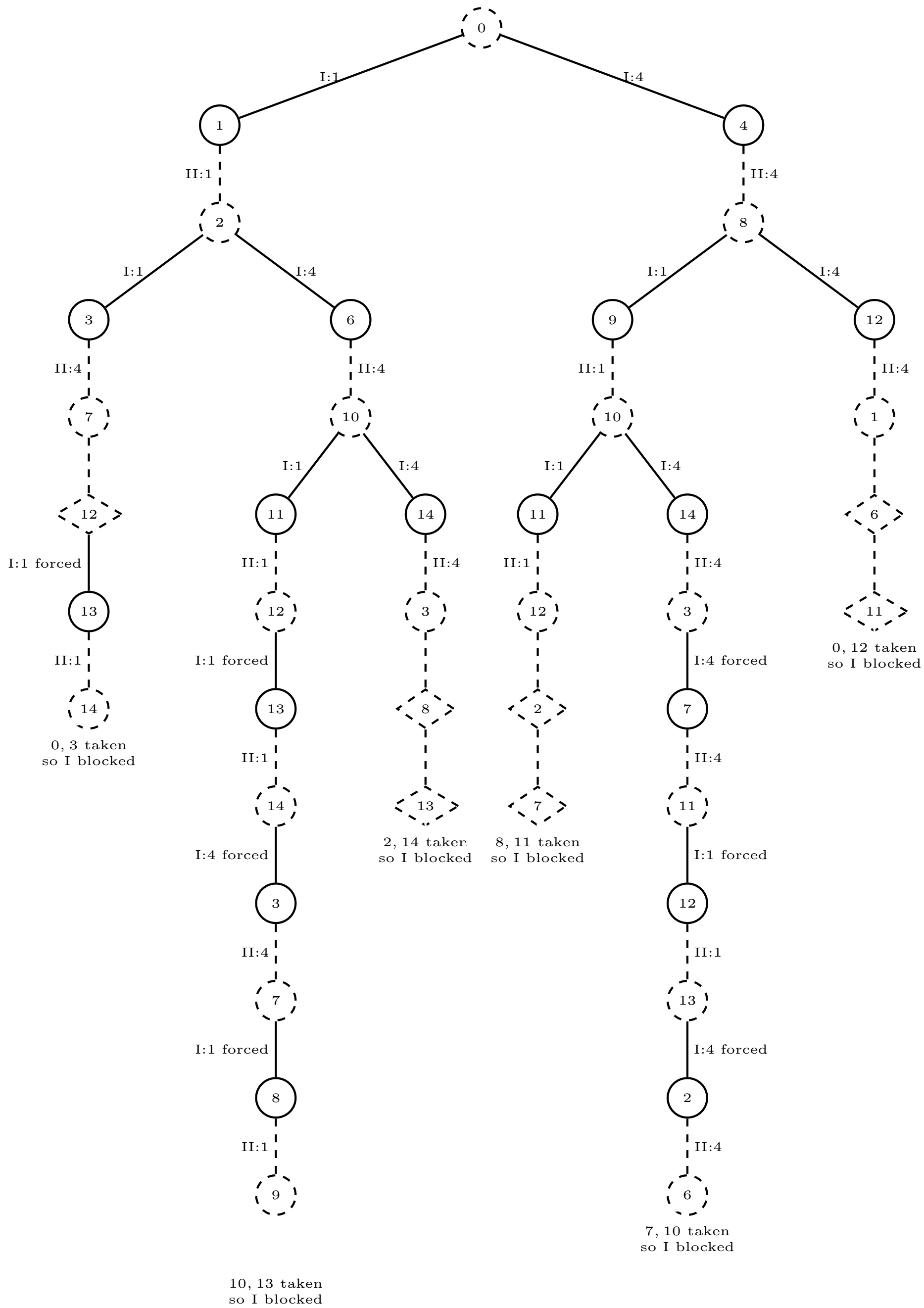

Figure 14: Winning strategy for Player II for $\Gamma(1, 4; 15)$.

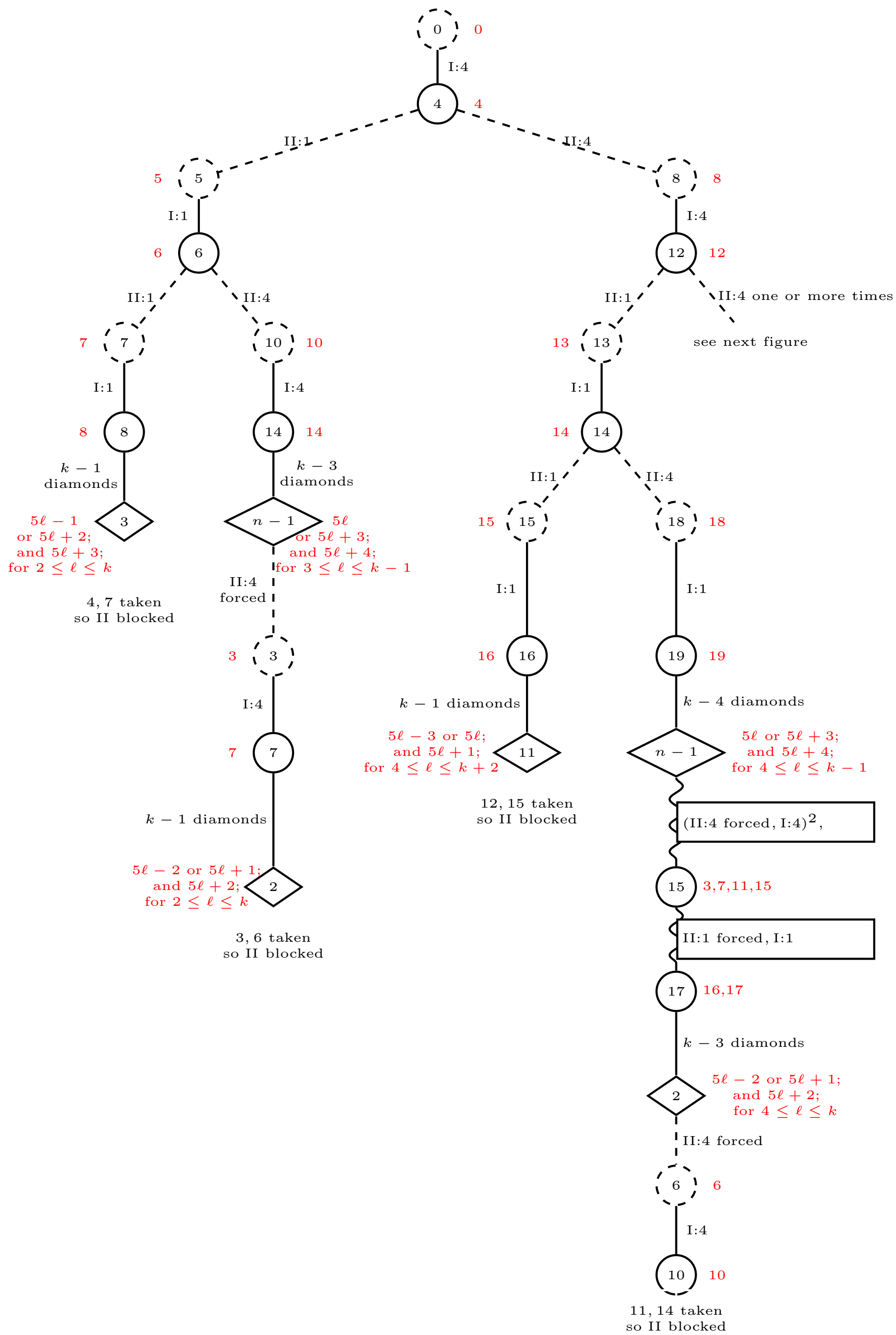

Figure 15: Winning strategy for Player I for $\Gamma(1, 4; 5k)$, where $k \geq 4$ and $n = 5k$.

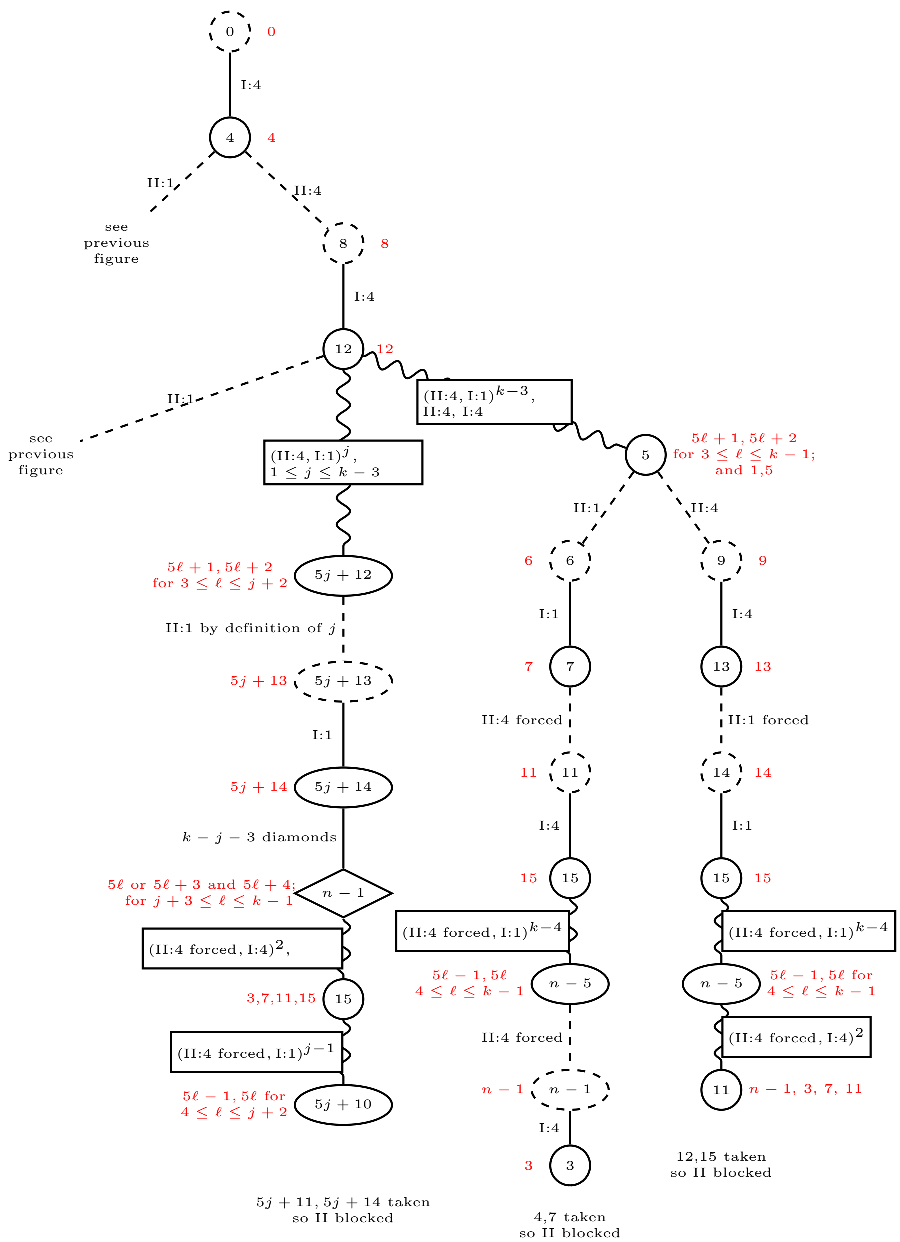

Figure 16: Winning strategy for Player I for $\Gamma(1,4;5k)$, where $k \geq 4$ and $n = 5k$.

4. Computational Database for Conjectures

We make a series of "primitive" conjectures in Section 4.1, and a host of other conjectures in Section 4.2, both meant to inspire new research about Kotzig's Nim. These conjectures were inspired by combing through the data generated from an exploratory computation and the resulting database. To assist the reader who is also interested in conjectures from exploratory data, we have posted all of the results of our computations on Kotzig's Nim. All of the computations were performed on an 8-core 3.0 GHz Apple MacPro, using `C++`. The database itself is located at `http://www.stat.purdue.edu/~mdw/kotzig/kotzigdatabase.txt` and has 185,134 entries at the time of submission. Each entry is one line of text, containing three positive integers m_1, m_2, n, and a character value N or P that classifies the starting location of the game of Kotzig's Nim. All triples $\{m_1, m_2; n\}$ with $1 \leq m_1 < m_2 \leq 72$ and $1 \leq n \leq 72$ are found in the database.

Some entries with $n > 72$ are also given for particular pairs m_1, m_2 for which $m_2 - m_1$ is relatively small, e.g.,, those pairs m_1, m_2 corresponding to Conjectures 4.1 through 4.6.

Note: In the opening sentence of the "Concluding Remarks" of [2], Fraenkel et al. make the conjecture that Player I can win $\Gamma(3, 4; n)$ for all $n = \pm 3 \bmod 7$. The data in our database verifies their conjecture for $n \leq 115$.

4.1. New Conjectures about Primitive Games

These six "primitive" conjectures are in the same spirit as Theorem 3.1, but we do not (yet) know of a proof for any of these six, and we would be delighted to correspond with readers who are interested in finding a methodology sufficiently robust to handle them. In each conjecture, the move set $M = \{m_1, m_2\}$ is fixed, and with $m_2 - m_1$ small. The initial positions of the games are classified as N-positions or P-positions, depending on the board length n.

Conjecture 4.1. If $n \in \{1, 3, 5, 7, 13, 15, 21, 23, 25, 33, 43\}$, or if $n \equiv 3 \bmod 10$ with $n \geq 83$, then $\Gamma(1, 5; n) \in \mathcal{P}$; otherwise, $\Gamma(1, 5; n) \in \mathcal{N}$. (Verified for $n \leq 140$.)

Conjecture 4.2. If

$$n \in \{1, 3, 5, 7, 11, 13, 19, 20, 21, 23, 25, 28, 30, 31, 33, 35, 38, 40, 41, 45, 48, 51, 52\},$$

or if n is equivalent to 2, 3, 5, or 6 modulo 7 with $n \geq 55$, then $\Gamma(2, 5; n) \in \mathcal{P}$; otherwise, $\Gamma(2, 5; n) \in \mathcal{N}$. (Verified for $n \leq 114$.)

Conjecture 4.3. If n is odd and $n \notin \{9, 11, 17\}$ then $\Gamma(3, 5; n) \in \mathcal{P}$; otherwise, $\Gamma(3, 5; n) \in \mathcal{N}$. (Verified for $n \leq 111$.)

Conjecture 4.4. If $n \in \{1, 3, 5, 7, 9, 13, 15, 19, 21, 23, 24, 25, 32, 33, 34, 39\}$, or if n is equivalent to 3, 4, 5, 6, or 7 modulo 9 with $n \geq 41$, then $\Gamma(4, 5; n) \in \mathcal{P}$; otherwise, $\Gamma(4, 5; n) \in \mathcal{N}$. (Verified for $n \leq 109$.)

Conjecture 4.5. If

$$n \in \{1, 3, 7, 11, 13, 18, 19, 21, 23, 29, 32, 33, 35, 37, 40, 41, 43, 44, 47, 48, 51, 54, 55\},$$

or if n is equivalent to 2, 3, 4, 7, or 10 modulo 11 with $n \geq 57$, then $\Gamma(4, 7; n) \in \mathcal{P}$; otherwise, $\Gamma(4, 7; n) \in \mathcal{N}$. (Verified for $n \leq 115$.)

Conjecture 4.6. If $n \in \{1, 3, 5, 7, 8, 9, 11, 13, 15, 17, 18, 20, 23, 25, 26, 27\}$, or if n is equivalent to 2, 3, 6, 7, or 9 modulo 11 with $n \geq 29$, or if $n \equiv 8 \bmod 11$ with $n \geq 74$, then $\Gamma(5, 6; n) \in \mathcal{P}$; otherwise, $\Gamma(5, 6; n) \in \mathcal{N}$. (Verified for $n \leq 112$.)

4.2. Other New Conjectures about Kotzig's Nim

Now we present several more conjectures about other types of periodicities in Kotzig's Nim.

The conjectures have been verified for all triples m_1, m_2, n found in the database generated from computations described in the introduction to Section 4.

4.2.1. A Conjecture Related to Primitive Games

In [2] and [1], $\Gamma(1, 2; n)$ and $\Gamma(2, 3; n)$, were already completely classified. Also $\Gamma(3, 4; n)$ was completely classified except for $n \equiv \pm 3 \bmod 7$.

Indeed, $\Gamma(m_1, m_1 + 1; n)$ was classified for:

1. $n = k(m_1 + m_2) - 1$ in Theorem 4 of [2],
2. $n = k(m_1 + m_2) + 0$ in Theorem 5 of [2],
3. $n = k(m_1 + m_2) + 1$ in Theorem 6 of [2].

The cases $m_2 = m_1 + 1$ and $n = k(m_1 + m_2) \pm 2$ remain open, but we introduce the following new conjecture:

Conjecture 4.7. Let $m_1 \geq 4$ and $m_2 = m_1 + 1$.

(1.) Consider $n = k(m_1 + m_2) - 2$.

If $k = 2$, then

$$\begin{aligned} &\Gamma\{m_1, m_2; k(m_1 + m_2) - 2\} \in \mathcal{P} \qquad \text{when } m_1 \equiv 2 \bmod 3, \\ &\Gamma\{m_1, m_2; k(m_1 + m_2) - 2\} \in \mathcal{N} \qquad \text{otherwise.} \end{aligned}$$

If $k = 1$ or $k \geq 3$, then

$$\begin{aligned} &\Gamma\{m_1, m_2; k(m_1 + m_2) - 2\} \in \mathcal{N} \qquad \text{when } m_1 \equiv 0 \bmod 3, \\ &\Gamma\{m_1, m_2; k(m_1 + m_2) - 2\} \in \mathcal{P} \qquad \text{otherwise.} \end{aligned}$$

(2.) Consider $n = k(m_1 + m_2) + 2$.

If $k = 0$ or $k = 2$, then

$$\Gamma\{m_1, m_2; k(m_1 + m_2) + 2\} \in \mathcal{N}.$$

If $k = 1$ or $k = 3$, then

$$\begin{aligned} &\Gamma\{m_1, m_2; k(m_1 + m_2) + 2\} \in \mathcal{N} && \text{when } m_1 \equiv 1 \bmod 3,\\ &\Gamma\{m_1, m_2; k(m_1 + m_2) + 2\} \in \mathcal{P} && \text{otherwise.}\end{aligned}$$

If $k \geq 4$, then

$$\begin{aligned} &\Gamma\{m_1, m_2; k(m_1 + m_2) + 2\} \in \mathcal{P} && \text{when } m_1 \equiv 2 \bmod 3,\\ &\Gamma\{m_1, m_2; k(m_1 + m_2) + 2\} \in \mathcal{N} && \text{otherwise.}\end{aligned}$$

Remark 4.8. Conjecture 4.7 holds for $m_1 \leq 3$ too, with only two exceptions:

$$\begin{aligned} \Gamma(1, 2; 3k - 2) &\in \mathcal{N} \qquad \text{for } k \geq 4;\\ \Gamma\{2, 3; 12\} &\in \mathcal{P}.\end{aligned}$$

but these relatively small cases were already known from the complete classifications of $\Gamma(1, 2; n)$, $\Gamma(2, 3; n)$, $\Gamma(3, 4; n)$. We explicitly wrote $m_1 \geq 4$ in Conjecture 4.7 to emphasize the new content.

4.2.2. A Conjecture for a Pair of Moves That Differ by 2

In Theorem 4 in [2], where $m_2 = m_1 + 1$, the games $\Gamma(m_1, m_2; k(m_1 + m_2) - 1)$ were classified as all being in $\mathcal{N}$.

Here we make an analogous new conjecture for $m_2 = m_1 + 2$:

Conjecture 4.9. Let $m_2 = m_1 + 2$. When m_1, m_2 are both odd, then

$$\Gamma\{m_1, m_1 + 2; k(m_1 + m_2) - 1\} \in \mathcal{P},$$

with one type of exception: When $m_1 = 1$, $m_2 = 3$, and $k \equiv 0 \bmod 3$, then for nonnegative a, we have $\Gamma = \{1, 3; 11 + 12a\} \in \mathcal{N}$.

Remark 4.10. We did not include the "even" situation in Conjecture 4.9, because those games are known to be in $\mathcal{N}$. Indeed, if m_1 and $m_2 = m_1 + 2$ are even, and if $n = k(m_1 + m_2) - 1$, then 2 is relatively prime to n. So by Lemma 2 of [2], it is equivalent to analyze $\Gamma(\frac{m_1}{2}, \frac{m_1+2}{2}; k(m_1 + m_2) - 1)$, which is in $\mathcal{N}$ by Theorem 4 of [2]. Thus $\Gamma(m_1, m_1 + 2; k(m_1 + m_2) - 1) \in \mathcal{N}$ too.

4.2.3. (Relatively) Low Hanging Fruit?

The following conjectures materialized after the first author spent some time combing through the data. A few special cases of these conjectures are already known, and we indicate some of these overlaps in the remarks below. On the other hand, in most cases, the conjectures below are disjoint from the known results.

Conjecture 4.11. For every pair m_1, m_2,

$$\Gamma\{m_1, m_2; 2(m_1 + m_2)\} \in \mathcal{N}.$$

Remark 4.12. When m_1 and m_2 are both odd, then—since n is even—Conjecture 4.11 follows from Theorem 3 of [2].

Conjecture 4.13. For every pair m_1, m_2,

$$\Gamma\{m_1, m_2; 2m_1\} \in \mathcal{N}.$$

Remark 4.14. Again, when m_1 and m_2 are both odd, then—since n is even—Conjecture 4.13 follows from Theorem 3 of [2].

Conjecture 4.15. Let $m_1 < m_2$. If there is a value k such that $\{m_1, m_2; 2m_2 - m_1\}$ equals $\{k, 2k, 3k\}$, $\{k, 4k, 7k\}$, $\{3k, 5k, 7k\}$, or $\{5k, 6k, 7k\}$, then $\Gamma\{m_1, m_2; 2m_2 - m_1\} \in \mathcal{P}$. Otherwise, $\Gamma\{m_1, m_2; 2m_2 - m_1\} \in \mathcal{N}$.

Remark 4.16. To prove Conjecture 4.15, it is safe to assume, without loss of generality, that m_1 and m_2 are relatively prime. For, if $g = \gcd(m_1, m_2)$, then we can write $m_1 = ga$ and $m_2 = gb$ where a, b are relatively prime, and then $\Gamma(m_1, m_2; 2m_2 - m_1)$ is equivalent to $\Gamma(a, b; 2b - a)$ by Lemma 1 of [2].

Conjecture 4.17. Let $m_2 = m_1 + d$ where $d \not\equiv 0 \bmod 3$. Then

$$\Gamma\{m_1, m_1 + d; 3(m_1 + m_2)\} \in \mathcal{N}.$$

Conjecture 4.18. Let $3 = m_1 < m_2$. If $m_2 \equiv 2 \bmod 4$, then $\Gamma\{3, m_2; m_2 + 1\} \in \mathcal{P}$; otherwise, $\Gamma\{3, m_2; m_2 + 1\} \in \mathcal{N}$.

Remark 4.19. Again, when m_1 and m_2 are both odd, then—since n is even—Conjecture 4.18 follows from Theorem 3 of [2].

In the following two conjectures (and also in Conjecture 4.18), move m_2 takes a player almost all the way around the circular board. Therefore, for large m_2, it might be more helpful to interpret a move of m_2 spaces clockwise as (equivalently) a move of $n - m_2$ spaces counterclockwise.

Conjecture 4.20. Let $1 = m_1 < m_2$. If m_2 is equivalent to 1, 5, or 6 modulo 10, then $\Gamma\{1, m_2; 2 + m_2\} \in \mathcal{P}$; otherwise, $\Gamma\{1, m_2; 2 + m_2\} \in \mathcal{N}$.

Conjecture 4.21. Let $2 = m_1 < m_2$. If m_2 is equivalent to 2, 6, or 7 modulo 10, then $\Gamma\{2, m_2; 1 + m_2\} \in \mathcal{P}$; otherwise, $\Gamma\{2, m_2; 1 + m_2\} \in \mathcal{N}$.

In the case where m_2 is even in Conjecture 4.21, then $\Gamma(2, m_2; 1+m_2)$ is equivalent to $\Gamma(1, \frac{m_2}{2}; 1 + m_2)$ by Lemma 2 of [2], so we get the following immediate corollary (that depends on the veracity of Conjecture 4.21, of course):

Corollary 4.22. *Let* $1 = m_1 < m_2$*. If* $m_2 \equiv 1 \bmod 5$ *or* $m_2 \equiv 3 \bmod 5$*, then* $\Gamma\{1, m_2; 1 + 2m_2\} \in \mathcal{P}$*; otherwise,* $\Gamma\{1, m_2; 1 + 2m_2\} \in \mathcal{N}$*.*

In this last conjecture (and also in Conjecture 4.13), one move takes a player approximately halfway around the circular board.

Conjecture 4.23. Let $2 = m_1 < m_2$. If m_2 is equivalent to 2, 3, or 6 modulo 10, then $\Gamma\{2, m_2; 2 + 2m_2\} \in \mathcal{P}$; otherwise, $\Gamma\{2, m_2; 2 + 2m_2\} \in \mathcal{N}$.

5. Conclusion

Kotzig's Nim is an excellent game for exploring with students. The game is easy to learn and fast to play, but investigations into the structure of the game require students to learn about a variety of computational and mathematical techniques. Many question remain unanswered. The authors hope that the present paper inspires new investigations. We invite others to utilize our database located at `http://www.stat.purdue.edu/~mdw/kotzig/kotzigdatabase.txt` and we urge our readers to continue working towards a complete classification of all Kotzig's Nim games with move set of size two, i.e., games of the form $\Gamma(m_1, m_2; n)$.

Acknowledgments The first author is thankful for the generosity of Kathryn and Art Lorenz through the Joseph Ruzicka Undergraduate Fellowship in the College of Science at Purdue University, which made this project possible. The second author's work is supported by the Center for Science of Information (CSoI), an NSF Science and Technology Center, under grant agreement CCF-0939370. We thank Richard K. Guy and Richard J. Nowakowski for their wonderful archive of unsolved problems [3], which piqued our curiosity to study Kotzig's Nim. We also thank an anonymous reviewer for detailed comments that improved the presentation and clarity of the paper.

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